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Formulation for Multiple Curved Crack Problem in a Finite Plate

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ABSTRACT

The formulation for the curved crack in a finite plate is established. The technique is the curved crack in a finite plate is divided into two subproblems i.e. the curved crack problem in an infinite plate and the finite plate without crack. For the first problem, the curved problem is formulated into Fredholm integral equation, where as for the second problem the complex boundary integral equations based on complex variables are considered. The solution of the coupled boundary integral equations gives the solution on the domain of the boundary.

Keywords: hypersingular integral equation, multiple cracks problem, curved cracks, finite plate.

1. Introduction

The multiple cracks problem in an infinite plate was modelled and investigated in earlier years. Panasyuk and Savruk (1977) suggested a singular integral equation for solving multiple cracks problem. Many researcher solved multiple cracks problem using singular, hypersingular or Fredholm integral equation for multiple straight, inclined or curved cracks problem (Chen, 1995a,b, Chen et al., 2003, Nik Long and Eshkuvator, 2009). However a few publications were devoted for modelling multiple crack problem in a finite plate. A general method for modelling multiple cracks problem in a arbitrary finite plate was studied by (Cheung et al., 1992). The efficient dual boundary element technique to obtain stress intensity factor at crack tips in a finite plate was discussed by (Chen, 1995b), and evaluating the T-stress for multiple cracks problem in finite region was done by (Chen et al., 2008). The principal of continues distribution of dislocation was used to model the curved crack problem in arbitrarily shaped finite plate by (Han and Dhanasekar, 2004). Karihaloo and Xiao (2001) expressed a hybrid crack element for the central crack problem in a finite plate. In the recent years, complex variable method presented by (Muskhelishvili, 1953) was used as a popular method to formulate crack problem in a finite plate. In addition, an alternating method depends on two type of integral equation was displayed for solving multiple cracks problem in a finite plate (Chen, 2011). One year later, couple boundary integral equation was applied to solve multiple cracks problem in a finite plate (Chen and Wang, 2012). In this paper, the formulation for the multiple curved crack problem in a finite plate is presented.

2. Analysis

The analysis presented in the following is based on two sub problem caused by the original problem. First, a curved crack in an infinite plate, and a system of Fredholm integral equation is suggested. The second, a finite region in the absence of crack where a boundary integral equation based on complex variable function method is used.(Fig.1)

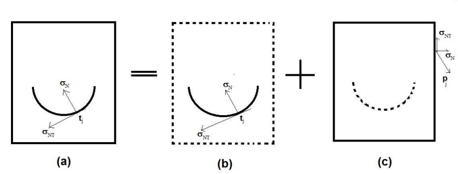

Figure 1: (a) The original problem, (b) A curve crack in infinite plate, (c) A finite plate with loading on the edges of plate

2.1 Analysis for multiple cracks problem in infinite plate using Fredholm integral equation

For a single crack problem in infinite plate, the loading on the crack face is denoted by

$$(\sigma_y - i\sigma_{xy})^{\pm} = P(t) - iQ(t), \quad |t| < a$$

Consider a point z on the crack with inclined angle β, the traction at point z can be written by

$$\sigma_N + i\sigma_{NT} = \frac{-1}{2\pi i}\int_{-a}^{a}[P(t) - iQ(t)][G(z,t) + \exp(-2i\beta)\overline{G(z,t)}]X(t)dt -$$

$$\frac{1}{2\pi i}\int_{-a}^{a}[P(t) + iQ(t)][(1 - \exp(-2i\beta))\overline{G(z,t)} + \exp(-2i\beta)(z - \overline{z})\overline{\acute{G}(z,t)}]X(t)dt \quad (1)$$

where

$$G(z,t) = \frac{1}{X(z)(z-t)}, \quad \acute{G}(z,t) = \frac{a^2 + tz - 2z^2}{(X(z))^3(z-t)^2}, \quad X(z) = \sqrt{z^2 - a^2}$$

It is noted that

$$\acute{G}(z,t) = \frac{dG(z,t)}{dt}$$

Now consider the multiple cracks in infinite plate shown in Fig.2. The multiple

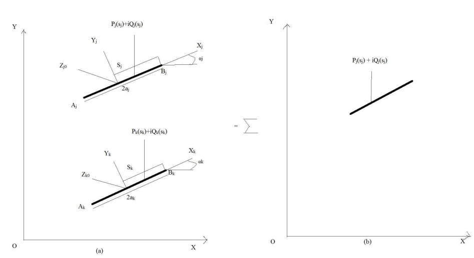

Figure 2: Superposition method for multiple cracks problem, (a) The original problem, (b) Superposition by many cracks with the undetermined traction

cracks problem can be considered as a superposition of N single crack with undetermined traction on crack. Using interaction among cracks and Eq. (1) a Fredholm integral equation is obtained as follows (Chen, 1995b, Chen et al., 2003):

$$P_k(s_k) - iQ_k(s_k) + \sum_{j=1}^{\prime N}\int_{-a_j}^{a_j}[P_j(s_j) - iQ_j(s_j)]C_{jk}(s_j, s_k)ds_j$$

$$+\sum_{j=1}^{\prime N}\int_{-a_j}^{a_j}[P_j(s_j)+iQ_j(s_j)]D_{jk}(s_j,s_k)ds_j = p_k(s_k)-iq_k(s_k), \quad (|s_k|<a_k, \quad k=1,...,N) \tag{2}$$

where

$$C_{jk}(s_j,s_k) = \frac{-X_j(s_j)}{2\pi i}[G_j(t_{jk},s_j)+\exp(2i(\alpha_j-\alpha_k))\overline{G_j(t_{jk},s_j)}]$$

$$D_{jk}(s_j,s_k) = \frac{-X_j(s_j)}{2\pi i}[(1-\exp(-2i(\alpha_j-\alpha_k)))\overline{G_j(t_{jk},s_j)}+\exp(2i(\alpha_j-\alpha_k))(t_{jk}-\overline{t_{jk}})\overline{\acute{G}_j(t_{jk},s_j)}]$$

$$G_j(z,s) = \frac{1}{X_j(z)(z-t)}, \acute{G}_j(z,s) = \frac{a_j^2+sz-2z^2}{(X_j(z))^3(z-t)^2}$$

$$t_{jk} = \exp(-i\alpha_j)(z_{k0}+s_k\exp(i\alpha_k)-z_{j0}), X_j(z) = \sqrt{z^2-a_j^2}$$

In Eq. (2), the prime in $\sum_{j=1}^{\prime N}$ means that the term $j=k$ should be excluded in the summation. The meaning of $\alpha_k, z_{k0}, s_k, a_k (k=1,2,....,N)$ has been indicated in Fig.2. The kernel $C_{jk}(s_j,s_k), D_{jk}(s_j,s_k)$ express the traction influence on the k^{th} crack caused by the traction applied on the j^{th} crack.

2.2 Analysis for the first boundary integral equation of interior problem

For solving the second part of original problem, a boundary integral equation for an interior region based on complex variable method takes the following form (Chen and Lin, 2010)

$$\frac{U(t_0)}{2} + B_1 i\int_\Gamma \frac{\kappa-1}{t-t_0}U(t)dt - B_1 i\int_\Gamma L_1(t,t_0)U(t)dt + B_1 i\int_\Gamma L_2(t,t_0)\overline{U(t)}dt$$
$$= B_2 i\int_\Gamma (2\kappa\ln|t-t_0|)Q(t)dt + B_2 i\int_\Gamma \frac{t-t_0}{\overline{t}-\overline{t_0}}\overline{Q(t)}dt \tag{3}$$

where

$$
\begin{aligned}
U(t) &= u(t) + iv(t) \\
Q(t) &= \sigma_N(t) + i\sigma_{NT}(t), \quad (t\epsilon\Gamma) \\
B_1 &= \frac{1}{2\pi(\kappa+1)} \\
B_2 &= \frac{1}{4\pi G(\kappa+1)} \\
L_1(t,t_0) &= \frac{d}{dt}\{ln\frac{t-t_0}{\overline{t}-\overline{t_0}}\} = -\frac{1}{t-t_0} + \frac{1}{\overline{t}-\overline{t_0}}\frac{\overline{dt}}{dt} \\
L_2(t,t_0) &= \frac{d}{dt}\{\frac{t-t_0}{\overline{t}-\overline{t_0}}\} = \frac{1}{\overline{t}-\overline{t_0}} - \frac{t-t_0}{(\overline{t}-\overline{t_0})^2}\frac{\overline{dt}}{dt}
\end{aligned}
$$

and G denotes the shear modulus of elasticity, $\kappa = 3 - 4\nu$ for plane strain problem when ν is Poisson's ratio. $Q(t) = \sigma_N(t) + i\sigma_{NT}(t)$ denotes the known traction applied on the boundary and $U(t) = u(t) + iv(t)$ denotes the unknown boundary displacements which will be archived after discretizing and solving the boundary integral equation (Eq. (3)). It is suppose that

$$
g'(t)|_{t=t(s)} = \frac{H(s)}{\sqrt{a^2-s^2}} = \frac{\sum_{j=1}^{N} \varphi_j T_j(s)}{\sqrt{a^2-s^2}} \quad Q(t)|_{t=t(s)} = \sum_{j=1}^{N} \psi_j T_j(s)
$$

After substituting $g'(t)$ and $Q(t)$, results

$$
\begin{aligned}
\frac{\sum_{j=1}^{N} \varphi_j T_j(s_0)}{2\sqrt{a^2-(s_0)^2}} &+ B_1 i\int_\Gamma \frac{(\kappa-1)\sum_{j=1}^{N}\varphi_j T_j(s)}{(t-t_0)\sqrt{a^2-s^2}}dt \\
-B_1 i\int_\Gamma L_1(t,t_0)\frac{\sum_{j=1}^{N} \varphi_j T_j(s)}{\sqrt{a^2-s^2}}dt &+ B_1 i\int_\Gamma L_2(t,t_0)\overline{\frac{\sum_{j=1}^{N} \varphi_j T_j(s)}{\sqrt{a^2-s^2}}}dt \\
= B_2 i\int_\Gamma (2\kappa\ln|t-t_0|)\sum_{j=1}^{N} \psi_j T_j(s)dt &+ B_2 i\int_\Gamma \frac{t-t_0}{\overline{t}-\overline{t_0}}\overline{\sum_{j=1}^{N} \psi_j T_j(s)}\overline{dt} \qquad (4)
\end{aligned}
$$

In order to solve Eq. (4), the first integral was multiplied by $\frac{(s-s_0)}{(s-s_0)}\frac{ds}{ds}$, two last integral was multiplied by $\frac{\sqrt{a^2-s^2}}{\sqrt{a^2-s^2}}\frac{ds}{ds}$ and the rest was multiplied by $\frac{ds}{ds}$. It is obtained

$$\frac{\sum_{j=1}^{N}\varphi_j T_j(s_0)}{2\sqrt{a^2-(s_0)^2}} + B_1 i(\kappa-1)\sum_{j=1}^{N}\varphi_j\int_{-a}^{a}\frac{A(s,s_0)}{(s-s_0)\sqrt{a^2-s^2}}ds$$
$$-B_1 i\sum_{j=1}^{N}\varphi_j\int_{-a}^{a}\frac{B(s,s_0)}{\sqrt{a^2-s^2}}ds + B_1 i\sum_{j=1}^{N}\varphi_j\int_{-a}^{a}\frac{C(s,s_0)}{\sqrt{a^2-s^2}}ds$$
$$= B_2 i2\kappa\sum_{j=1}^{N}\psi_j\int_{-a}^{a}\frac{D(s,s_0)}{\sqrt{a^2-s^2}}ds + B_2 i\sum_{j=1}^{N}\psi_j\int_{-a}^{a}\frac{E(s,s_0)}{\sqrt{a^2-s^2}}ds \tag{5}$$

where

$$A(s,s_0)=T_j(s)\frac{(s-s_0)}{(t-t_0)}\frac{dt}{ds} \quad ; \quad B(s,s_0)=T_j(s)L_1(t,t_0)\frac{dt}{ds}$$
$$C(s,s_0)=\overline{T_j(s)}L_2(t,t_0)\frac{dt}{ds} \quad ; \quad D(s,s_0)=T_j(s)\ln|t-t_0|\frac{\sqrt{a^2-s^2}dt}{ds}$$

$$E(s,s_0)=\overline{T_j(s)}\frac{t-t_0}{\overline{t}-\overline{t_0}}\sqrt{a^2-s^2}\frac{\overline{dt}}{ds}$$

Applying quadrature rules, Eq. (5) is turned to

$$\frac{\sum_{j=1}^{N}\varphi_j T_j(s_0)}{2\sqrt{a^2-(s_0)^2}} + B_1 i(\kappa-1)\frac{\pi}{Q}\sum_{j=1}^{N}\varphi_j\sum_{q=1}^{Q}\frac{A(s_q,s_0)}{(s_q-s_0)} -$$
$$B_1 i\frac{\pi}{Q}\sum_{j=1}^{N}\varphi_j\sum_{q=1}^{Q}B(s_q,s_0) + B_1 i\frac{\pi}{Q}\sum_{j=1}^{N}\overline{\varphi_j}\sum_{q=1}^{Q}C(s_q,s_0) =$$
$$B_2 i2\kappa\frac{\pi}{Q}\sum_{j=1}^{N}\psi_j\sum_{q=1}^{Q}D(s_q,s_0) + B_2 i\frac{\pi}{Q}\sum_{j=1}^{N}\overline{\psi_j}\sum_{q=1}^{Q}E(s_q,s_0) \tag{6}$$

where

$$A(s_q,s_0)=T_j(s_q)\frac{(s_q-s_0)}{(t_q-t_0)}\frac{dt_q}{ds_q} \quad ; \quad B(s_q,s_0)=T_j(s_q)L_1(t_q,t_0)\frac{dt_q}{ds_q}$$
$$C(s_q,s_0)=\overline{T_j(s_q)}L_2(t_q,t_0)\frac{dt_q}{ds_q} \quad ; \quad D(s_q,s_0)=T_j(s_q)\ln|t_q-t_0|\frac{\sqrt{a^2-s_q^2}dt_q}{ds_q}$$

$$E(s_q,s_0)=\overline{T_j(s_q)}\frac{t_q-t_0}{\overline{t_q}-\overline{t_0}}\frac{\sqrt{a^2-s_q^2}\,\overline{dt_q}}{ds_q}$$

multiplying both side in $\int_{-a}^{a} U_{k-1}(s_0)\sqrt{a^2-s_0^2}\,ds_0$ and using quadrature rule, it is given

$$
\begin{aligned}
&B_1 i(\kappa-1)\frac{\pi}{Q}\sum_{j=1}^{N}\varphi_j\sum_{p=1}^{P}\sum_{q=1}^{Q}\frac{A(s_q,s_{0p})}{(s_q-s_{0p})}U_{k-1}(s_{0p})\frac{\pi\sin^2\beta_p}{P+1} \quad - \\
&B_1 i\frac{\pi}{Q}\sum_{j=1}^{N}\varphi_j\sum_{p=1}^{P}\sum_{q=1}^{Q}B(s_q,s_{0p})U_{k-1}(s_{0p})\frac{\pi\sin^2\beta_p}{P+1} \quad + \\
&B_1 i\frac{\pi}{Q}\sum_{j=1}^{N}\overline{\varphi_j}\sum_{p=1}^{P}\sum_{q=1}^{Q}C(s_q,s_{0p})U_{k-1}(s_{0p})\frac{\pi\sin^2\beta_p}{P+1} \quad = \\
&B_2 i2\kappa\sum_{j=1}^{N}\psi_j\sum_{p=1}^{P}\sum_{q=1}^{Q}D(s_q,s_{0p})U_{k-1}(s_{0p})\frac{\pi\sin^2\beta_p}{P+1} \quad + \\
&B_2 i\sum_{j=1}^{N}\overline{\psi_j}\sum_{p=1}^{P}\sum_{q=1}^{Q}E(s_q,s_{0p})U_{k-1}(s_{0p})\frac{\pi\sin^2\beta_p}{P+1} \quad .
\end{aligned} \tag{7}
$$

Also, The first integral in Eq. (6) will be expressed as

$$
\begin{aligned}
&\frac{1}{4}\sum_{j=1}^{N}\varphi_j\left(\sum_{p=1}^{P}\frac{\cos((j+k)\beta_p)}{j+k}+\sum_{p=1}^{P}\frac{\cos((k-j)\beta_p)}{k-j}\right), \qquad k\geq j \\
&\frac{1}{4}\sum_{j=1}^{N}\varphi_j\left(\sum_{p=1}^{P}\frac{\cos((j+k)\beta_p)}{j+k}+\sum_{p=1}^{P}\frac{\cos((j-k)\beta_p)}{j-k}\right), \qquad k\leq j-1
\end{aligned}
$$

2.3 Analysis for the second boundary integral equation of interior problem

The traction at inner point τ can be evaluated by second boundary integral equation (Chen and Lin, 2010).

$$
\begin{aligned}
\frac{1}{2G}Q(\tau) &= -2B_1 i\int_\Gamma \frac{1}{(t-\tau)^2}U(t)dt - B_1 i\int_\Gamma M_1(t,\tau)U(t)dt \\
&+ B_1 i\int_\Gamma M_2(t,\tau)\overline{U(t)}dt + B_2 i\int_\Gamma \frac{\kappa-1}{(t-\tau)}Q(t)dt \\
&+ B_2 i\int_\Gamma \kappa K_1(t,\tau)Q(t)dt + B_2 i\int_\Gamma K_2(t,\tau)\overline{Q(t)dt} \quad (\tau\epsilon S^+)
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
K_1(t,\tau) &= \frac{d}{dt}\{ln\frac{t-\tau}{\bar{t}-\overline{\tau}}\} = -\frac{1}{t-\tau} + \frac{1}{\bar{t}-\overline{\tau}}\frac{\overline{d\tau}}{d\tau} \\
K_2(t,\tau) &= -\frac{d}{dt}\{\frac{t-\tau}{\bar{t}-\overline{\tau}}\} = \frac{1}{\bar{t}-\overline{\tau}} - \frac{(t-\tau)}{(\bar{t}-\overline{\tau})^2}\frac{\overline{d\tau}}{d\tau} \\
M_1(t,\tau) &= \frac{-d}{dt}\{K_1(t,\tau)\} = \frac{-d}{d\tau}\{\frac{d}{dt}\{ln\frac{t-\tau}{\bar{t}-\overline{\tau}}\}\} \qquad (9) \\
&= \frac{-1}{(t-\tau)^2} + \frac{1}{(\bar{t}-\overline{\tau})^2}\frac{\overline{dt}}{dt}\frac{\overline{d\tau}}{d\tau} \\
M_2(t,\tau) &= -\frac{d}{dt}\{K_2(t,\tau)\} = \frac{d}{d\tau}\{\frac{d}{dt}\{\frac{t-\tau}{\bar{t}-\overline{\tau}}\} \\
&= \frac{1}{(\bar{t}-\overline{\tau})^2}(\frac{\overline{dt}}{dt} + \frac{\overline{d\tau}}{d\tau}) - 2\frac{(t-\tau)}{(\bar{t}-\overline{\tau})^3}\frac{\overline{dt}}{dt}\frac{\overline{d\tau}}{d\tau}
\end{aligned}
$$

In above equation t means a point on the boundary and $\bar{t}$ is conjugate of t respectively, dt and $\overline{dt}$ represent the tangent of boundary and conjugate of dt, respectively.
more over τ, $\overline{\tau}$ are the point on the crack and its conjugate and $d\tau$ and $\overline{d\tau}$ is same as dt and $\overline{dt}$.
After discritization of Eq. (3) and using quadrature rule, Eq. (8) will be presented by

$$
\begin{aligned}
\frac{1}{2G}Q(\tau_k) &= -2B_1 i\frac{\pi}{N}\sum_{j=1}^{N} A_2(s_j,\tau_k)H(s_j) \\
-B_1 i\frac{\pi}{N}\sum_{j=1}^{N} B_2(s_j,\tau_k)H(s_j) &+ B_1 i\frac{\pi}{N}\sum_{j=1}^{N} C_2(s_j,\tau_k)H(s_j) \\
+B_2 i\frac{\pi}{N}\sum_{j=1}^{N} D_2(s_j,\tau_k)F(s_j) &+ B_2 i\frac{\pi}{N}\sum_{j=1}^{N} E_2(s_j,\tau_k)F(s_j) \\
&+ B_2 i\frac{\pi}{N}\sum_{j=1}^{N} G_2(s_j,\tau_k)\overline{F(s_j)} \ \ (\tau_k \epsilon S^+) \qquad (10)
\end{aligned}
$$

where

$$\begin{aligned} A_2(s_j,\tau_k) &= \frac{1}{(t_j-\tau_k)^2}\frac{dt_j}{ds_j} \quad ; \quad B_2(s_j,\tau_k) = M_1(t_j,\tau_k)\frac{dt_j}{ds_j} \\ C_2(s_j,\tau_k) &= M_2(t_j,\tau_k)\frac{dt_j}{ds_j} \quad ; \quad D_2(s_j,\tau_k) = \frac{\kappa-1}{(t_j-\tau_k)}\frac{dt_j}{ds_j} \\ E_2(s_j,\tau_k) &= \kappa K_1(t_j,\tau_k)\frac{dt_j}{ds_j} \quad ; \quad G_2(s_j,\tau_k) = K_2(t_j,\tau_k)\frac{\overline{dt_j}}{ds_j} \end{aligned} \tag{11}$$

At the end, it will converted as $[Q_\tau] = [A_\tau] * \{H(s)\} + [G_\tau] * \{F(s)\}$ which $Q_\tau = \sigma_N(\tau) + i\sigma_{NT}(\tau)$ means that traction on the crack face. After substituting displacement (it is obtained from Eq. (7)) and traction at boundary point in Eq. (10), the traction at an inner point τ of plate will be evaluated.

After substituting Q_τ as the traction of inner points in the right hand side of Eq. (2), $P - iQ$ at inner point τ will be obtained. Then we can get the stress intensity factor at crack tips.

3. Solution strategy

In solving Eq. (3) and Eq. (8), the following quadrature rules will be used:

$$\int_{-a}^{a} \frac{g(s)ds}{(s-s_0)\sqrt{a^2-s^2}}ds = \frac{\pi}{M}\sum_{j=1}^{M}\frac{g(s_j)}{(s_j-s_{0k})} \quad j=1,...,M, k=1,...,M-1 \tag{12}$$

$$s_j = a\cos(\frac{(2j-1)\pi}{2M}) \quad s_{ok} = a\cos(\frac{k\pi}{M})$$

$$\int_{-a}^{a} f(t)(a^2-t^2)^{-1/2}dt = \sum_{q=1}^{Q} w_q f(t_q), \quad w_q = \frac{\pi}{Q}, \quad t_q = a\cos(\frac{(2q-1)\pi}{2Q}) \tag{13}$$

and

$$\int_{-a}^{a} f(t)(a^2-t^2)^{1/2}dt = \sum_{p=1}^{P} w_p f(t_p), \quad w_q = \frac{\pi(\sin^2(\beta_p))}{P+1}, \quad t_p = a\cos[\frac{p\pi}{P+1}] \tag{14}$$

and relation between first and second Chebyshev polynomial

$$T_j(x)U_k(x) = \frac{1}{2}(U_{j+k}(x) + U_{k-j}(x)), \qquad k \geq j-1$$
$$T_j(x)U_k(x) = \frac{1}{2}(U_{j+k}(x) + U_{j-k-2}(x)), \qquad k \leq j-2 \qquad (15)$$

it is known that $T_q(t) = T_q(\cos(\theta)) = \cos(q\theta)$ is Chebyshev polynomial of first kind and $U_p(t) = U_p(\cos\beta) = \dfrac{\sin[(P+1)\beta]}{\sin(\beta)}$ is Chebyshev polynomial of second kind.

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